Étale structures in countable model theory and descriptive set theory

Ronnie Chen

University of Michigan

August 22, 2023

Background

Let $\mathcal{L}$ be a countable first-order language.

Background

Let $\mathcal{L}$ be a countable first-order language.

The "**space of countable $\mathcal{L}$-structures**" is:

$$X := \prod_{n\text{-ary rel } R \in \mathcal{L}} 2^{\mathbb{N}^n} \times \prod_{n\text{-ary fn } f \in \mathcal{L}} \mathbb{N}^{\mathbb{N}^n}$$

Background

Let $\mathcal{L}$ be a countable first-order language.

The "**space of countable $\mathcal{L}$-structures**" is:

$$X := \prod_{n\text{-ary rel } R \in \mathcal{L}} 2^{\mathbb{N}^n} \times \prod_{n\text{-ary fn } f \in \mathcal{L}} \mathbb{N}^{\mathbb{N}^n}$$

or:

$$X := \bigsqcup_{N \leq \mathbb{N}} \left(\prod_{n\text{-ary rel } R \in \mathcal{L}} 2^{N^n} \times \prod_{n\text{-ary fn } f \in \mathcal{L}} N^{N^n} \right)$$

Background

Let $\mathcal{L}$ be a countable first-order language.

The "**space of countable $\mathcal{L}$-structures**" is:

$$X := \prod_{n\text{-ary rel } R \in \mathcal{L}} 2^{\mathbb{N}^n} \times \prod_{n\text{-ary fn } f \in \mathcal{L}} \mathbb{N}^{\mathbb{N}^n}$$

or:

$$X := \bigsqcup_{N \leq \mathbb{N}} \left(\prod_{n\text{-ary rel } R \in \mathcal{L}} 2^{N^n} \times \prod_{n\text{-ary fn } f \in \mathcal{L}} N^{N^n} \right)$$

$$X := \bigsqcup_{N \subseteq \mathbb{N}} \left(\prod_{n\text{-ary rel } R \in \mathcal{L}} 2^{N^n} \times \prod_{n\text{-ary fn } f \in \mathcal{L}} N^{N^n} \right) \quad (\text{topology on } \subseteq \mathbb{N}?)$$

Background

Let $\mathcal{L}$ be a countable first-order language.

The "**space of countable $\mathcal{L}$-structures**" is:

$$X := \prod_{n\text{-ary rel } R \in \mathcal{L}} 2^{\mathbb{N}^n} \times \prod_{n\text{-ary fn } f \in \mathcal{L}} \mathbb{N}^{\mathbb{N}^n}$$

or:

$$X := \bigsqcup_{N \leq \mathbb{N}} \left(\prod_{n\text{-ary rel } R \in \mathcal{L}} 2^{N^n} \times \prod_{n\text{-ary fn } f \in \mathcal{L}} N^{N^n} \right)$$

$$X := \bigsqcup_{N \subseteq \mathbb{N}} \left(\prod_{n\text{-ary rel } R \in \mathcal{L}} 2^{N^n} \times \prod_{n\text{-ary fn } f \in \mathcal{L}} N^{N^n} \right) \quad \text{(topology on } \subseteq \mathbb{N}\text{?)}$$

or (for groups):

$$X := \{K \subseteq \langle \mathbb{N} \rangle \mid K \text{ is a normal subgroup}\}$$

Background

Let $\mathcal{L}$ be a countable first-order language.

The "**space of countable $\mathcal{L}$-structures**" is:

$$X := \prod_{n\text{-ary rel } R \in \mathcal{L}} 2^{\mathbb{N}^n} \times \prod_{n\text{-ary fn } f \in \mathcal{L}} \mathbb{N}^{\mathbb{N}^n}$$

or:

$$X := \bigsqcup_{N \leq \mathbb{N}} \left(\prod_{n\text{-ary rel } R \in \mathcal{L}} 2^{N^n} \times \prod_{n\text{-ary fn } f \in \mathcal{L}} N^{N^n} \right)$$

$$X := \bigsqcup_{N \subseteq \mathbb{N}} \left(\prod_{n\text{-ary rel } R \in \mathcal{L}} 2^{N^n} \times \prod_{n\text{-ary fn } f \in \mathcal{L}} N^{N^n} \right) \quad \text{(topology on } \subseteq \mathbb{N}\text{?)}$$

or (for groups):

$$X := \{K \subseteq \langle \mathbb{N} \rangle \mid K \text{ is a normal subgroup}\}$$

These are spaces of **codes** for $\mathcal{L}$-structures, not $\mathcal{L}$-structures themselves!

$$X = \{\text{codes for } \mathcal{L}\text{-structures}\}$$

$$\big\downarrow\!\!\downarrow$$

$$\{\text{all } \mathcal{L}\text{-structures}\}$$

$X = \{\text{codes for } \mathcal{L}\text{-structures}\}$: a nice (e.g., Polish) topol. space

$$\Big\downarrow \text{“cts open”}$$

$\{\text{all } \mathcal{L}\text{-structures}\}$: ???

Étale structures

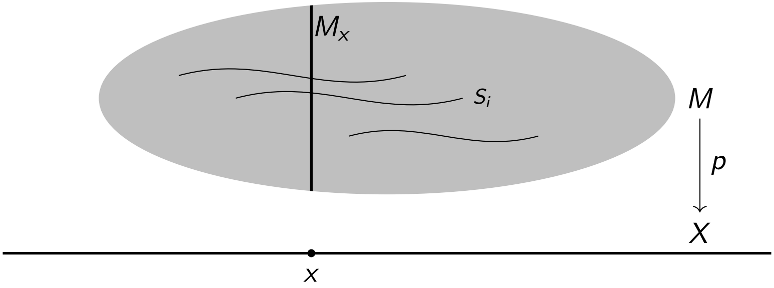
Let X be a topological space.

Étale structures

Let X be a topological space.

Definition An **étale space over** X is a topological space M equipped with a continuous map $p : M \to X$ (the "projection") which is a **local homeomorphism**:

- $M = \bigcup_i S_i$ for **open sections** $S_i \subseteq M$ s.t. $p|S_i : S_i \to X$ is an open embedding.
  "The fibers $M_x := p^{-1}(x)$ are discrete, continuously in x."

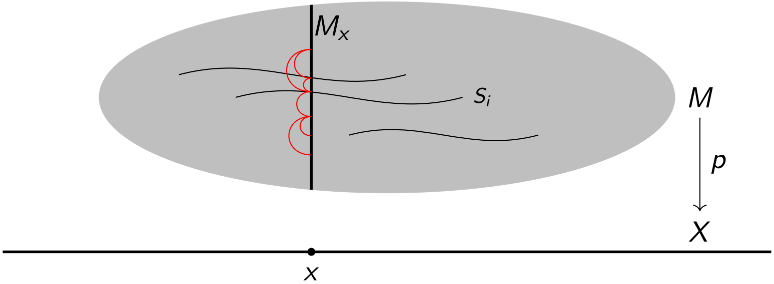


Étale structures

Let X be a topological space, $\mathcal{L}$ be a first-order language.

Definition An **étale $\mathcal{L}$-structure** $\mathcal{M}$ **over** X is:

- an underlying étale space $p : M \to X$;
- for each n-ary function symbol $f \in \mathcal{L}$, a continuous map $f^{\mathcal{M}} : M_X^n \to M$ over X;
- for each n-ary relation symbol $R \in \mathcal{L}$, an open set $R^{\mathcal{M}} \subseteq M_X^n$.



Examples of étale structures

- $$\begin{array}{c} M = X \times \mathbb{N} \\ \Big\downarrow \mathsf{proj}_1 \\ X = \prod_{n\text{-ary } R \in \mathcal{L}} 2^{\mathbb{N}^n} \times \prod_{n\text{-ary } f \in \mathcal{L}} \mathbb{N}^{\mathbb{N}^n} \end{array}$$

Examples of étale structures

- $$\begin{array}{c} M = X \times \mathbb{N} \\ \Big\downarrow \text{proj}_1 \\ X = \prod_{n\text{-ary } R \in \mathcal{L}} 2^{\mathbb{N}^n} \times \prod_{n\text{-ary } f \in \mathcal{L}} \mathbb{N}^{\mathbb{N}^n} \end{array}$$
- $$\begin{array}{c} M = \{(x = (N, \dots), a) \in X \times \mathbb{N} \mid a \in N\} \\ \Big\downarrow \text{proj}_1 \\ X = \bigsqcup_{N \leq \mathbb{N}} (\prod_{n\text{-ary } R \in \mathcal{L}} 2^{N^n} \times \prod_{n\text{-ary } f \in \mathcal{L}} N^{N^n}) \end{array}$$

Examples of étale structures

- $$\begin{array}{c} M = X \times \mathbb{N} \\ \Big\downarrow \text{proj}_1 \\ X = \prod_{n\text{-ary } R \in \mathcal{L}} 2^{\mathbb{N}^n} \times \prod_{n\text{-ary } f \in \mathcal{L}} \mathbb{N}^{\mathbb{N}^n} \end{array}$$
- $$\begin{array}{c} M = \{(x = (N, \dots), a) \in X \times \mathbb{N} \mid a \in N\} \\ \Big\downarrow \text{proj}_1 \\ X = \bigsqcup_{N \leq \mathbb{N}} (\prod_{n\text{-ary } R \in \mathcal{L}} 2^{N^n} \times \prod_{n\text{-ary } f \in \mathcal{L}} N^{N^n}) \end{array}$$
- same but with one-pt cptification topology on $\overline{\mathbb{N}} = \{0, 1, 2, \dots, \mathbb{N}\}$

Examples of étale structures

- $$\begin{array}{c} M = X \times \mathbb{N} \\ \big\downarrow \mathrm{proj}_1 \\ X = \prod_{n\text{-ary } R\in\mathcal{L}} 2^{\mathbb{N}^n} \times \prod_{n\text{-ary } f\in\mathcal{L}} \mathbb{N}^{\mathbb{N}^n} \end{array}$$
- $$\begin{array}{c} M = \{(x = (N, \dots), a) \in X \times \mathbb{N} \mid a \in N\} \\ \big\downarrow \mathrm{proj}_1 \\ X = \coprod_{N\leq\mathbb{N}} (\prod_{n\text{-ary } R\in\mathcal{L}} 2^{N^n} \times \prod_{n\text{-ary } f\in\mathcal{L}} N^{N^n}) \end{array}$$
- same but with one-pt cptification topology on $\overline{\mathbb{N}} = \{0, 1, 2, \dots, \mathbb{N}\}$
- Assume $\mathcal{L}$ is functional.
$$\begin{array}{c} M = \{(\sim, a) \mid \sim \in X,\ a \in \langle\mathbb{N}\rangle/\sim\} \quad (\text{quotient of } X \times \langle\mathbb{N}\rangle) \\ \big\downarrow \mathrm{proj}_1 \\ X = \{\sim \subseteq \langle\mathbb{N}\rangle^2 = \{\mathcal{L}\text{-terms over } \mathbb{N}\}^2 \mid \sim \text{ is a congruence}\} \end{array}$$

(. . . , Artin–Grothendieck–Verdier ’71, Makkai–Reyes ’77, Joyal–Tierney ’84, . . .)

topology	étale model theory
cts map $f : X \to Y =$ “all $\mathcal{L}$-strs”	étale structure $\mathcal{M} \to X$

(..., Artin–Grothendieck–Verdier '71, Makkai–Reyes '77, Joyal–Tierney '84, ...)

topology	étale model theory
cts map $f : X \to Y =$ "all $\mathcal{L}$-strs"	étale structure $\mathcal{M} \to X$
cts open $f : X \to Y$	

(. . . , Artin–Grothendieck–Verdier ’71, Makkai–Reyes ’77, Joyal–Tierney ’84, . . .)

topology	étale model theory
cts map $f : X \to Y =$ “all $\mathcal{L}$-strs”	étale structure $\mathcal{M} \to X$
cts open $f : X \to Y$	
$\ker(f) = \{(x_1, x_2) \in X^2 \mid f(x_1) = f(x_2)\}$	$\mathrm{Iso}(\mathcal{M}) = \{(x_1, x_2, g) \mid g : \mathcal{M}_{x_1} \cong \mathcal{M}_{x_2}\}$ $=$ **isomorphism groupoid** of $\mathcal{M}$

(..., Artin–Grothendieck–Verdier '71, Makkai–Reyes '77, Joyal–Tierney '84, ...)

topology	étale model theory
cts map $f : X \to Y =$ "all $\mathcal{L}$-strs"	étale structure $\mathcal{M} \to X$
cts open $f : X \to Y$ (onto image)	open $U \subseteq M_X^n$ has Σ_1 **saturation** $\mathrm{Iso}(\mathcal{M}) \cdot U = \phi^{\mathcal{M}}$ (ϕ uses $\wedge, \bigvee, \exists$)
$\ker(f) = \{(x_1, x_2) \in X^2 \mid f(x_1) = f(x_2)\}$	$\mathrm{Iso}(\mathcal{M}) = \{(x_1, x_2, g) \mid g : \mathcal{M}_{x_1} \cong \mathcal{M}_{x_2}\}$ $=$ **isomorphism groupoid** of $\mathcal{M}$

Examples of étale structures

- $$\begin{array}{c} M = X \times \mathbb{N} \\ \big\downarrow \mathrm{proj}_1 \\ X = \prod_{n\text{-ary } R \in \mathcal{L}} 2^{\mathbb{N}^n} \times \prod_{n\text{-ary } f \in \mathcal{L}} \mathbb{N}^{\mathbb{N}^n} \end{array}$$

  $\mathrm{Iso}(\mathcal{M}) \cdot \{(x, \vec{a}) \mid \phi^x(\vec{b})\}$ (ϕ uses $\wedge, \neg$)
  $= (\exists \vec{z}\,(\phi(\vec{z}) \wedge \text{“}(\vec{y}, \vec{z}) \cong (\vec{a}, \vec{b})\text{”}))^{\mathcal{M}}$
  $\rightsquigarrow \Sigma_2$ (Σ_1 after Morleyizing $\neg$atomic)

- $$\begin{array}{c} M = \{(x = (N, \dots), a) \in X \times \mathbb{N} \mid a \in N\} \\ \big\downarrow \mathrm{proj}_1 \\ X = \bigsqcup_{N \le \mathbb{N}} (\prod_{n\text{-ary } R \in \mathcal{L}} 2^{N^n} \times \prod_{n\text{-ary } f \in \mathcal{L}} N^{N^n}) \end{array}$$

- same but with one-pt cptification topology on $\overline{\mathbb{N}} = \{0, 1, 2, \dots, \mathbb{N}\}$
- Assume $\mathcal{L}$ is functional.

  $$\begin{array}{c} M = \{(\sim, a) \mid \sim \in X,\ a \in \langle \mathbb{N} \rangle / \sim\} \quad (\text{quotient of } X \times \langle \mathbb{N} \rangle) \\ \big\downarrow \mathrm{proj}_1 \\ X = \{\sim \subseteq \langle \mathbb{N} \rangle^2 = \{\mathcal{L}\text{-terms over } \mathbb{N}\}^2 \mid \sim \text{ is a congruence}\} \end{array}$$

Examples of étale structures

- $$\begin{array}{c} M = X \times \mathbb{N} \\ \big\downarrow \text{proj}_1 \\ X = \prod_{n\text{-ary } R \in \mathcal{L}} 2^{\mathbb{N}^n} \times \prod_{n\text{-ary } f \in \mathcal{L}} \mathbb{N}^{\mathbb{N}^n} \end{array}$$

  $\mathrm{Iso}(\mathcal{M}) \cdot \{(x, \vec{a}) \mid \phi^x(\vec{b})\}$ (ϕ uses $\wedge, \neg$)
  $= (\exists \vec{z}\,(\phi(\vec{z}) \wedge \text{“}(\vec{y}, \vec{z}) \cong (\vec{a}, \vec{b})\text{”}))^{\mathcal{M}}$
  $\rightsquigarrow \Sigma_2$ (Σ_1 after Morleyizing $\neg$atomic)

- $$\begin{array}{c} M = \{(x = (N, \dots), a) \in X \times \mathbb{N} \mid a \in N\} \\ \big\downarrow \text{proj}_1 \\ X = \bigsqcup_{N \le \mathbb{N}} (\prod_{n\text{-ary } R \in \mathcal{L}} 2^{N^n} \times \prod_{n\text{-ary } f \in \mathcal{L}} N^{N^n}) \end{array}$$

  $\exists x_0 \cdots x_{n-1} \bigwedge_{i \neq j} (x_i \neq x_j)$
  $\rightsquigarrow \Sigma_4$ saturations

- same but with one-pt cptification topology on $\overline{\mathbb{N}} = \{0, 1, 2, \dots, \mathbb{N}\}$ $\rightsquigarrow \Sigma_3$
- Assume $\mathcal{L}$ is functional.

  $$\begin{array}{c} M = \{(\sim, a) \mid \sim\, \in X,\ a \in \langle \mathbb{N} \rangle / \sim\} \quad (\text{quotient of } X \times \langle \mathbb{N} \rangle) \\ \big\downarrow \text{proj}_1 \\ X = \{\sim\, \subseteq \langle \mathbb{N} \rangle^2 = \{\mathcal{L}\text{-terms over } \mathbb{N}\}^2 \mid\, \sim \text{ is a congruence}\} \end{array}$$

  $\rightsquigarrow \Sigma_1$

topology	étale model theory
cts map $f : X \to Y =$ "all $\mathcal{L}$-strs"	étale structure $\mathcal{M} \to X$
cts open $f : X \to Y$ (onto image)	open $U \subseteq M^n_X$ has Σ_1 saturation
$\ker(f) = \{(x_1, x_2) \in X^2 \mid f(x_1) = f(x_2)\}$	$\mathrm{Iso}(\mathcal{M}) = \{(x_1, x_2, g) \mid g : \mathcal{M}_{x_1} \cong \mathcal{M}_{x_2}\}$
Polish $Y =$ open quot of $\mathbb{N}^\mathbb{N}$	Π_2 theory $\mathcal{T} \Rightarrow$ 2nd-ctbl étale $\mathcal{M} \to \mathbb{N}^\mathbb{N}$ w/ Σ_1 sat s.t. $\mathrm{Mod}(\mathcal{T}) = [\{\mathcal{M}_x\}_x]_\cong$

topology	étale model theory
cts map $f : X \to Y =$ "all $\mathcal{L}$-strs"	étale structure $\mathcal{M} \to X$
cts open $f : X \to Y$ (onto image)	open $U \subseteq M^n_X$ has Σ_1 saturation
$\ker(f) = \{(x_1, x_2) \in X^2 \mid f(x_1) = f(x_2)\}$	$\mathrm{Iso}(\mathcal{M}) = \{(x_1, x_2, g) \mid g : \mathcal{M}_{x_1} \cong \mathcal{M}_{x_2}\}$
Polish $Y =$ open quot of $\mathbb{N}^\mathbb{N}$	Π_2 theory $\mathcal{T} \Rightarrow$ 2nd-ctbl étale $\mathcal{M} \to \mathbb{N}^\mathbb{N}$ w/ Σ_1 sat s.t. $\mathrm{Mod}(\mathcal{T}) = [\{\mathcal{M}_x\}_x]_\cong$
	2nd-ctbl étale $\mathcal{M}$ w/ Σ_1 sat $\Rightarrow$ Π_2-ax'ble

Theorem (C. 2023) *For every second-countable étale structure $\mathcal{M}$ with Σ_1 saturations over a (quasi-)Polish X, the collection of fibers $\mathcal{M}_x$ is Π_2-axiomatizable.*

topology	étale model theory
cts map $f : X \to Y =$ "all $\mathcal{L}$-strs"	étale structure $\mathcal{M} \to X$
cts open $f : X \to Y$ (onto image)	open $U \subseteq M^n_X$ has Σ_1 saturation
$\ker(f) = \{(x_1, x_2) \in X^2 \mid f(x_1) = f(x_2)\}$	$\mathrm{Iso}(\mathcal{M}) = \{(x_1, x_2, g) \mid g : \mathcal{M}_{x_1} \cong \mathcal{M}_{x_2}\}$
Polish $Y =$ open quot of $\mathbb{N}^{\mathbb{N}}$	Π_2 theory $\mathcal{T} \Rightarrow$ 2nd-ctbl étale $\mathcal{M} \to \mathbb{N}^{\mathbb{N}}$ w/ Σ_1 sat s.t. $\mathrm{Mod}(\mathcal{T}) = [\{\mathcal{M}_x\}_x]_{\cong}$
cts open f onto image $\Rightarrow$ image is Π^0_2	2nd-ctbl étale $\mathcal{M}$ w/ Σ_1 sat $\Rightarrow$ Π_2-ax'ble

Theorem (C. 2023) *For every second-countable étale structure $\mathcal{M}$ with Σ_1 saturations over a (quasi-)Polish X, the collection of fibers $\mathcal{M}_x$ is Π_2-axiomatizable.*

Proof is a combination of:
(Sierpinski, de Brecht) Open quotient of (q-)Polish is (q-)Polish.
(Alexandrov, de Brecht) (Q-)Polish subspace of second-countable is Π^0_2.
(Ryll-Nardzewski, Suzuki) Atomic models are Π_2-categorical.

topology	étale model theory
cts map $f : X \to Y =$ "all $\mathcal{L}$-strs"	étale structure $\mathcal{M} \to X$
cts open $f : X \to Y$ (onto image)	open $U \subseteq M_X^n$ has Σ_1 saturation
$\ker(f) = \{(x_1, x_2) \in X^2 \mid f(x_1) = f(x_2)\}$	$\mathrm{Iso}(\mathcal{M}) = \{(x_1, x_2, g) \mid g : \mathcal{M}_{x_1} \cong \mathcal{M}_{x_2}\}$
Polish $Y =$ open quot of $\mathbb{N}^\mathbb{N}$	Π_2 theory $\mathcal{T} \Rightarrow$ 2nd-ctbl étale $\mathcal{M} \to \mathbb{N}^\mathbb{N}$ w/ Σ_1 sat s.t. $\mathrm{Mod}(\mathcal{T}) = [\{\mathcal{M}_x\}_x]_\cong$
cts open f onto image $\Rightarrow$ image is Π^0_2	2nd-ctbl étale $\mathcal{M}$ w/ Σ_1 sat $\Rightarrow$ Π_2-ax'ble
$\exists^*_f(A) = \{y \mid \exists^* x \in f^{-1}(y)\,(x \in A)\}$	

topology	étale model theory
cts map $f : X \to Y =$ "all $\mathcal{L}$-strs"	étale structure $\mathcal{M} \to X$
cts open $f : X \to Y$ (onto image)	open $U \subseteq M_X^n$ has Σ_1 saturation
$\ker(f) = \{(x_1, x_2) \in X^2 \mid f(x_1) = f(x_2)\}$	$\mathrm{Iso}(\mathcal{M}) = \{(x_1, x_2, g) \mid g : \mathcal{M}_{x_1} \cong \mathcal{M}_{x_2}\}$
Polish $Y =$ open quot of $\mathbb{N}^\mathbb{N}$	Π_2 theory $\mathcal{T} \Rightarrow$ 2nd-ctbl étale $\mathcal{M} \to \mathbb{N}^\mathbb{N}$ w/ Σ_1 sat s.t. $\mathrm{Mod}(\mathcal{T}) = [\{\mathcal{M}_x\}_x]_\cong$
cts open f onto image $\Rightarrow$ image is Π^0_2	2nd-ctbl étale $\mathcal{M}$ w/ Σ_1 sat $\Rightarrow$ Π_2-ax'ble
$\exists^*_f(A) = \{y \mid \exists^* x \in f^{-1}(y)\,(x \in A)\}$ $\Rightarrow$ f-invariant Σ^0_α $A \in f^{-1}(\Sigma^0_\alpha)$	

topology	étale model theory
cts map $f : X \to Y =$ "all $\mathcal{L}$-strs"	étale structure $\mathcal{M} \to X$
cts open $f : X \to Y$ (onto image)	open $U \subseteq M_X^n$ has Σ_1 saturation
$\ker(f) = \{(x_1, x_2) \in X^2 \mid f(x_1) = f(x_2)\}$	$\mathrm{Iso}(\mathcal{M}) = \{(x_1, x_2, g) \mid g : \mathcal{M}_{x_1} \cong \mathcal{M}_{x_2}\}$
Polish $Y =$ open quot of $\mathbb{N}^{\mathbb{N}}$	Π_2 theory $\mathcal{T} \Rightarrow$ 2nd-ctbl étale $\mathcal{M} \to \mathbb{N}^{\mathbb{N}}$ w/ Σ_1 sat s.t. $\mathrm{Mod}(\mathcal{T}) = [\{\mathcal{M}_x\}_x]_{\cong}$
cts open f onto image $\Rightarrow$ image is Π_2^0	2nd-ctbl étale $\mathcal{M}$ w/ Σ_1 sat $\Rightarrow$ Π_2-ax'ble
$\exists_f^*(A) = \{y \mid \exists^* x \in f^{-1}(y)\,(x \in A)\}$ $\Rightarrow$ f-invariant Σ_α^0 $A \in f^{-1}(\Sigma_\alpha^0)$	$A^\triangle = \{\vec{a} \in M_x^n \mid \exists^*(y, x, g) \in \mathrm{Iso}(\mathcal{M})\,(\vec{a} \in gA)\}$

topology	étale model theory
cts map $f : X \to Y =$ "all $\mathcal{L}$-strs"	étale structure $\mathcal{M} \to X$
cts open $f : X \to Y$ (onto image)	open $U \subseteq M_X^n$ has Σ_1 saturation
$\ker(f) = \{(x_1, x_2) \in X^2 \mid f(x_1) = f(x_2)\}$	$\mathrm{Iso}(\mathcal{M}) = \{(x_1, x_2, g) \mid g : \mathcal{M}_{x_1} \cong \mathcal{M}_{x_2}\}$
Polish $Y =$ open quot of $\mathbb{N}^{\mathbb{N}}$	Π_2 theory $\mathcal{T} \Rightarrow$ 2nd-ctbl étale $\mathcal{M} \to \mathbb{N}^{\mathbb{N}}$ w/ Σ_1 sat s.t. $\mathrm{Mod}(\mathcal{T}) = [\{\mathcal{M}_x\}_x]_{\cong}$
cts open f onto image $\Rightarrow$ image is Π_2^0	2nd-ctbl étale $\mathcal{M}$ w/ Σ_1 sat $\Rightarrow$ Π_2-ax'ble
$\exists_f^*(A) = \{y \mid \exists^* x \in f^{-1}(y)\,(x \in A)\}$ $\Rightarrow$ f-invariant Σ_α^0 $A \in f^{-1}(\Sigma_\alpha^0)$	$A^{\triangle} = \{\vec{a} \in M_x^n \mid \exists^*(y, x, g) \in \mathrm{Iso}(\mathcal{M})\,(\vec{a} \in gA)\}$ $\Rightarrow$ $\cong$-invariant Σ_α^0 $A = \phi^{\mathcal{M}}$ for Σ_α ϕ

Theorem (C. 2023) *For every second-countable étale structure $\mathcal{M}$ with Σ_1 saturations over a (quasi-)Polish X, every $\cong$-invariant Σ_α^0 $A \subseteq M_X^n$ is definable by a Σ_α ϕ.*

This includes both the classical Lopez-Escobar theorem and a recent version for "positive" formulas due to (Bazhenov–Fokina–Rossegger–Soskova–Vatev '23).

topology	étale model theory
cts map $f : X \to Y =$ "all $\mathcal{L}$-strs"	étale structure $\mathcal{M} \to X$
cts open $f : X \to Y$ (onto image)	open $U \subseteq M_X^n$ has Σ_1 saturation
$\ker(f) = \{(x_1, x_2) \in X^2 \mid f(x_1) = f(x_2)\}$	$\mathrm{Iso}(\mathcal{M}) = \{(x_1, x_2, g) \mid g : \mathcal{M}_{x_1} \cong \mathcal{M}_{x_2}\}$
Polish $Y =$ open quot of $\mathbb{N}^\mathbb{N}$	Π_2 theory $\mathcal{T} \Rightarrow$ 2nd-ctbl étale $\mathcal{M} \to \mathbb{N}^\mathbb{N}$ w/ Σ_1 sat s.t. $\mathrm{Mod}(\mathcal{T}) = [\{\mathcal{M}_x\}_x]_\cong$
cts open f onto image $\Rightarrow$ image is Π^0_2	2nd-ctbl étale $\mathcal{M}$ w/ Σ_1 sat $\Rightarrow$ Π_2-ax'ble
$\exists^*_f(A) = \{y \mid \exists^* x \in f^{-1}(y)\,(x \in A)\}$ $\Rightarrow$ f-invariant Σ^0_α $A \in f^{-1}(\Sigma^0_\alpha)$	$A^\triangle = \{\vec{a} \in M^n_x \mid \exists^*(y, x, g) \in \mathrm{Iso}(\mathcal{M})\,(\vec{a} \in gA)\}$ $\Rightarrow$ $\cong$-invariant Σ^0_α $A = \phi^\mathcal{M}$ for Σ_α ϕ
Baire category theorem	omitting types theorem

topology	étale model theory
cts map $f : X \to Y =$ "all $\mathcal{L}$-strs"	étale structure $\mathcal{M} \to X$
cts open $f : X \to Y$ (onto image)	open $U \subseteq M_X^n$ has Σ_1 saturation
$\ker(f) = \{(x_1, x_2) \in X^2 \mid f(x_1) = f(x_2)\}$	$\mathrm{Iso}(\mathcal{M}) = \{(x_1, x_2, g) \mid g : \mathcal{M}_{x_1} \cong \mathcal{M}_{x_2}\}$
Polish $Y =$ open quot of $\mathbb{N}^\mathbb{N}$	Π_2 theory $\mathcal{T} \Rightarrow$ 2nd-ctbl étale $\mathcal{M} \to \mathbb{N}^\mathbb{N}$ w/ Σ_1 sat s.t. $\mathrm{Mod}(\mathcal{T}) = [\{\mathcal{M}_x\}_x]_\cong$
cts open f onto image $\Rightarrow$ image is Π^0_2	2nd-ctbl étale $\mathcal{M}$ w/ Σ_1 sat $\Rightarrow$ Π_2-ax'ble
$\exists^*_f(A) = \{y \mid \exists^* x \in f^{-1}(y)\,(x \in A)\}$ $\Rightarrow$ f-invariant Σ^0_α $A \in f^{-1}(\Sigma^0_\alpha)$	$A^\triangle = \{\vec{a} \in M_x^n \mid \exists^*(y, x, g) \in \mathrm{Iso}(\mathcal{M})\,(\vec{a} \in gA)\}$ $\Rightarrow$ $\cong$-invariant Σ^0_α $A = \phi^\mathcal{M}$ for Σ_α ϕ
Baire category theorem	omitting types theorem
Y has quotient topology	(Joyal–Tierney) $\mathcal{T}$ determined up to bi-interp.

Speculations on continuous logic

A **metric structure** has an underlying complete metric space in place of an underlying set, and (Lipschitz) $[0, 1]$-valued instead of 2-valued predicates.

Speculations on continuous logic

A **metric structure** has an underlying complete metric space in place of an underlying set, and (Lipschitz) $[0, 1]$-valued instead of 2-valued predicates.

Unlike in discrete logic, there is no obvious single canonical "space of structures":

- structures on Urysohn sphere $\mathbb{U}$ (GK, EFPRTT, IM, CL)
- structures on closed subspaces of $\mathbb{U}$
- structures on completions of $\mathbb{N}$ w/ pseudometric (BDNT, HMT)

Speculations on continuous logic

A **metric structure** has an underlying complete metric space in place of an underlying set, and (Lipschitz) $[0, 1]$-valued instead of 2-valued predicates.

Unlike in discrete logic, there is no obvious single canonical "space of structures":

- structures on Urysohn sphere $\mathbb{U}$ (GK, EFPRTT, IM, CL)
- structures on closed subspaces of $\mathbb{U}$
- structures on completions of $\mathbb{N}$ w/ pseudometric (BDNT, HMT)

A **metric-étale space** $p : M \to X$ is a "topometric bundle" whose fiberwise metrics $d_x : M_x^2 \to \mathbb{R}$ "induce the topology on M_x, uniformly in x".

Speculations on continuous logic

A **metric structure** has an underlying complete metric space in place of an underlying set, and (Lipschitz) $[0,1]$-valued instead of 2-valued predicates.

Unlike in discrete logic, there is no obvious single canonical "space of structures":

- structures on Urysohn sphere $\mathbb{U}$ (GK, EFPRTT, IM, CL)
- structures on closed subspaces of $\mathbb{U}$
- structures on completions of $\mathbb{N}$ w/ pseudometric (BDNT, HMT)

A **metric-étale space** $p : M \to X$ is a "topometric bundle" whose fiberwise metrics $d_x : M_x^2 \to \mathbb{R}$ "induce the topology on M_x, uniformly in x".

We can extend large parts of the dictionary to metric-étale structures.
However, we are hampered by a lack of a pre-existing "continuous topos theory"; in particular, the Joyal–Tierney theorem seems tricky.